

Lower bound on the mean square displacement of particles in the hard disk model

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Abstract

The hard disk model is a 2D Gibbsian process of particles interacting via pure hard core repulsion. At high particle density the model is believed to show orientational order, however, it is known not to exhibit positional order. Here we investigate to what extent particle positions may fluctuate. We consider a finite volume version of the model in a box of dimensions $2n \times 2n$ with arbitrary boundary configuration, and we show that the mean square displacement of particles near the center of the box is bounded from below by $c \log n$. The result generalizes to a large class of models with fairly arbitrary interaction.

Key words: Hard disk model, Gibbsian point processes, 2D crystallization, absence of positional order, fluctuations of positions, mean square displacement, pure hard core repulsion, percolation.

1 Introduction

Crystallization is an important topic in equilibrium state statistical physics. At low temperature or at high particle density particles arrange themselves into regular patterns such as lattice-like structures and thus form a state which is usually referred to as solid. It is not always clear to which extent such a solid indeed resembles a lattice. In particular for 2D solids this has been debated for a long time. A first model to be studied was the 2D harmonic crystal. Peierls showed that in this model particles are not localized (see [P1], [P2]), i.e. with increasing size of the system fluctuations of particle positions grow unboundedly. More precisely the mean square displacement of a particle from its ideal lattice position is of order $\log n$ if n is the size of the system. The absence of positional order was shown to be a general feature of 2D particle systems. This follows from ideas of Mermin and Wagner ([MW] and [M]) and was first applied to particle positions in a continuum setting by Fröhlich and Pfister ([FP1] and [FP2]). In spite of this negative result, 2D solids are believed to exist, and indeed their lattice-like structure may be due to orientational order rather than positional order. So far this could not be shown rigorously for any realistic particle system, but this is supported by results obtained from simulations (e.g. see [BK]) and results in related models with a predefined lattice structure used to label particles (see [MR], [HMR], [G] and [Au]).

A particularly simple model with the above properties is the hard disk model. Here the interaction between particles is a pure hard-core repulsion, i.e. any two point particles are forced to keep a distance of $> 2r$ but do not interact otherwise (see [L] for a review of properties of this model). Equivalently particles can be thought of as disks with radius r and the interaction prevents disks from overlapping. Besides r the only parameter of the model is the activity z regulating the particle density. The model can be obtained from a Poisson point process with intensity z by conditioning on particles to keep distance $> 2r$. Due to its simplicity it is a good starting point for investigations, and it has been studied extensively through simulations. According to these simulations the model exhibits three different phases (see [BK]):

- small z : liquid (or gas) phase. The model does not exhibit any order and we have exponential decay of positional and orientational correlations.
- intermediate z : hexatic phase. The model exhibits orientational order and exponential decay of positional correlations.
- large z : solid phase. The model exhibits orientational order and algebraic decay of positional correlations.

The phase transition from liquid to hexatic is of first order and the phase transition from hexatic to solid is continuous. In contrast to this detailed picture, not much is known rigorously:

- In [R1] the result of Fröhlich and Pfister is extended to the hard disk model, i.e. it is shown that there is no positional order for any value of z .
- In [Ar] a percolation result is obtained: Suppose that for given $\epsilon > 0$ any two disks of distance $\leq \epsilon$ are connected. It should be expected that in the high density regime (i.e. for z sufficiently large) we have percolation of connected disks. This is established for $\epsilon > r$.

In the result presented in this paper we refine the result of [R1] and show that the hard disk model shows the same behaviour as the harmonic crystal, in that the mean square displacement of a particle from its ideal lattice position is at least of order $\log n$ if n is the size of the system. The formulation of such a result is not straightforward: Unlike in the harmonic crystal, in the hard disk model there is no a priori labelling of particles that would allow to pinpoint a specific particle and investigate the fluctuations of its position. Instead, we will describe the fluctuations of positions in terms of a certain transformation of particle configurations within a box of size $2n \times 2n$ with the following properties:

- Particles near the center of the box are translated by an amount of order $\sqrt{\log n}$, whereas particles near the boundary are kept fixed.
- Locally the transformation almost preserves the relative position of particles. In particular the hard core condition is not violated.
- The transformation only has a mild impact on the probability measure describing the hard disk model.

Our main theorem shows that a transformation with these properties can be constructed for the hard disk model, and by means of a corollary we will explain why this transformation should be thought of as providing a lower bound of order $\log n$ for the mean square displacement of particles. We note that our results are not restricted to pure hard core repulsion but can be extended to fairly arbitrary interactions.

A variant of the transformation described above was the main tool used in [R1] for showing the absence of positional order. In [MP] the method was adjusted to a lattice setting and used to show a delocalization result for the random Lipschitz surface model, including a lower bound on fluctuations of order $\log n$. We use improvements and refinements of the method from [MP] and adjust them back to the continuous setting of the hard disk model. Some arguments are taken straight from [R1] but repeated here for the sake of completeness.

In Section 2 we give a precise description of our result (theorem and corollary) and we outline to what extent it can be generalized. In Section 3 we give a proof of the corollary. In Section 4 we give a proof of the theorem. All technical parts and lemmas used in this proof are relegated to Section 5.

2 Result

Before explaining the hard disk model we describe the general setting. On the single particle state space \mathbb{R}^2 we consider the Borel- σ -algebra denoted by \mathcal{B}^2 and the Lebesgue-measure denoted by λ^2 . When integrating with respect to λ^2 we use the usual abbreviation $dx := d\lambda^2(x)$. Our set of particle configurations is

$$\mathcal{X} := \{X \subset \mathbb{R}^2 : \#X_\Lambda < \infty \text{ for every bounded } \Lambda \subset \mathbb{R}^2\},$$

the set of all locally finite subsets of the plane. Here $X_\Lambda := X \cap \Lambda$ denotes the restriction of a configuration X to a set Λ and $\#$ denotes the cardinality of a set. Let $\mathcal{X}_\Lambda := \{X \in \mathcal{X} : X \subset \Lambda\}$ denote the set of all configurations in $\Lambda \subset \mathbb{R}^2$. The σ -algebras \mathcal{F} and \mathcal{F}_Λ on \mathcal{X} and \mathcal{X}_Λ respectively are defined to be generated by the counting variables $N_{\Lambda'}$ ($\Lambda' \in \mathcal{B}^2$), where $N_{\Lambda'}(X) := \#X_{\Lambda'}$. Our reference measure on $(\mathcal{X}_\Lambda, \mathcal{F}_\Lambda)$ for a bounded set $\Lambda \in \mathcal{B}^2$ is the distribution ν_Λ of the Poisson point process. We have

$$\int \nu_\Lambda(dX) f(X) = e^{-\lambda^2(\Lambda)} \sum_{k \geq 0} \frac{1}{k!} \int_{\Lambda^k} dx_1 \dots dx_k f(\{x_i : 1 \leq i \leq k\}),$$

for any \mathcal{F}_Λ -measurable function $f : \mathcal{X}_\Lambda \rightarrow \mathbb{R}_+$. For convenience we would like to include the boundary configuration into the reference measure. Here any configuration $Y \in \mathcal{X}$ can serve as a boundary configuration, and the reference measure with this boundary configuration is denoted by $\nu_\Lambda(\cdot|Y)$. It should be thought of producing a Poisson point process inside Λ and the deterministic configuration Y_{Λ^c} outside Λ , i.e.

$$\int \nu_\Lambda(dX|Y) f(X) = \int \nu_\Lambda(dX) f(X_\Lambda \cup Y_{\Lambda^c})$$

for any \mathcal{F} -measurable function $f : \mathcal{X} \rightarrow \mathbb{R}_+$. $\nu_\Lambda(\cdot|Y)$ can be considered as a probability measure on $(\mathcal{X}, \mathcal{F})$ or as a probability measure on $(\mathcal{X}_\Lambda, \mathcal{F}_\Lambda)$.

For the definition of the hard disk model we need to take the hard core into account. By rescaling we may assume that the disk diameter equals 1. The hard core can be built into the setting by restricting the configuration space to the set of hard core configurations

$$\mathcal{X}^h := \{X \subset \mathbb{R}^2 : \forall x, y \in X : x \neq y \Rightarrow |x - y|_2 > 1\},$$

where $|\cdot|_2$ denotes Euclidean distance. Alternatively this hard core repulsion can be modelled by the two-body interaction $U : (\mathbb{R}^2)^2 \rightarrow \mathbb{R} \cup \{\infty\}$:

$$U(x, y) := \begin{cases} \infty & \text{for } |x - y|_2 \leq 1 \\ 0 & \text{for } |x - y|_2 > 1. \end{cases}$$

The a priori particle density is modelled by an activity parameter $z > 0$ corresponding to the chemical potential $-\log z$ of the system. We note that the inverse temperature, which usually serves as a second parameter for a model of this type, does not play a role in case of pure hard core repulsion. The hard disk model can now be described in terms of the finite volume Gibbs distributions $\mu_\Lambda^z(\cdot|Y)$ in volume $\Lambda \in \mathcal{B}^2$ (bounded) with respect to the boundary configuration $Y \in \mathcal{X}^h$. $\mu_\Lambda^z(\cdot|Y)$ is a probability measure defined by

$$\mu_\Lambda^z(dX|Y) = \frac{1}{Z_\Lambda^z(Y)} e^{-H_\Lambda(X)} z^{\#X_\Lambda} \nu_\Lambda(dX|Y) = \frac{1}{Z_\Lambda^z(Y)} 1_{\mathcal{X}^h}(X) z^{\#X_\Lambda} \nu_\Lambda(dX|Y).$$

Here

$$H_\Lambda(X) := \frac{1}{2} \sum_{x \neq y \in X_\Lambda} U(x, y) + \sum_{x \in X_\Lambda, y \in X_{\Lambda^c}} U(x, y)$$

denotes the Hamiltonian and $Z_\Lambda^z(Y)$ denotes the partition function which plays the role of a normalizing constant. $\mu_\Lambda^z(\cdot|Y)$ can be interpreted as a Poisson point process in Λ with intensity z conditioned on the event that any two points in Λ keep a distance of ≥ 1 and any point in Λ keeps a distance of ≥ 1 to Y_{Λ^c} . In this paper we only work in finite volume, but we would like to note that the model can also be extended to infinite volume by means of the Dobrushin-Lanford-Ruelle (DLR) equation: Any probability measure μ on $(\mathcal{X}, \mathcal{F})$ that is compatible with the above finite volume Gibbs distributions for arbitrary Λ and Y is called an infinite volume Gibbs measure at activity z .

Our aim is to investigate the extent to which particles in typical configurations produced by $\mu_\Lambda^z(\cdot|Y)$ deviate from a lattice structure in terms of positional order. For sake of simplicity we only consider domains of the form

$$\Lambda_n := [-n, n]^2 \subset \mathbb{R}^2 \quad (n \in \mathbb{N}).$$

Consequently we will use the abbreviations $\mu_n^z := \mu_{\Lambda_n}^z$, $Z_n^z := Z_{\Lambda_n}^z$, etc. The unit vector $e \in \mathbb{R}^2$ will be used for modelling the direction of the proposed deviation. For definiteness we will only consider the direction

$$e := e_1 = (1, 0).$$

Our main result is the following:

Theorem 1 *Let $z > 0$, $\epsilon := \min\{\frac{1}{48z}, \frac{1}{4}\}$, $\delta \in (0, \frac{1}{2}]$ and $n \geq 200$. There is a transformation $\mathfrak{T}_n : \mathcal{X} \rightarrow \mathcal{X}$ of the form $\mathfrak{T}_n(X) = \{x + t_{n,X}(x)e_1 : x \in X\}$ with $t_{n,X} : \mathbb{R}^2 \rightarrow [0, \infty)$ and there is a set of good configurations $G_n \in \mathcal{F}$ such that:*

- (1) *For all $X \in \mathcal{X}$ and $x \notin \Lambda_n$ we have $t_{n,X}(x) = 0$.*
- (2) *For all $X \in G_n$ and $x \in X_{\Lambda_{\sqrt{n}}}$ we have $t_{n,X}(x) = \delta\epsilon\sqrt{\log n}$.*
- (3) *For all $X \in \mathcal{X}$ and $x, y \in X$ we have $|t_{n,X}(x) - t_{n,X}(y)| \leq \delta|x - y|_2$.*
- (4) *For all $Y \in \mathcal{X}^h$ we have $\mu_n^z(G_n^c|Y) \leq \frac{1}{n}$.*
- (5) *\mathfrak{T}_n is bijective and $\mathfrak{T}_n(\mathcal{X}^h) = \mathcal{X}^h$.*
- (6) *For every $Y \in \mathcal{X}^h$ $\mu_n^z(\cdot|Y)$ has a density $\varphi_n : \mathcal{X} \rightarrow \mathbb{R}$ w.r.t. $\mu_n^z(\cdot|Y) \circ \mathfrak{T}_n^{-1}$.*
- (7) *For every $Y \in \mathcal{X}^h$ we have $\mu_n^z(|\log(\varphi_n \bar{\varphi}_n)| |Y) \leq 120\delta^2$.*

In addition, the transformation $\bar{\mathfrak{T}}_n : \mathcal{X} \rightarrow \mathcal{X}$, $\bar{\mathfrak{T}}_n(X) = \{x - t_{n,X}(x)e_1 : x \in X\}$ has properties analogous to (5) and (6). The function $\bar{\varphi}_n$ appearing in (7) denotes the corresponding density.

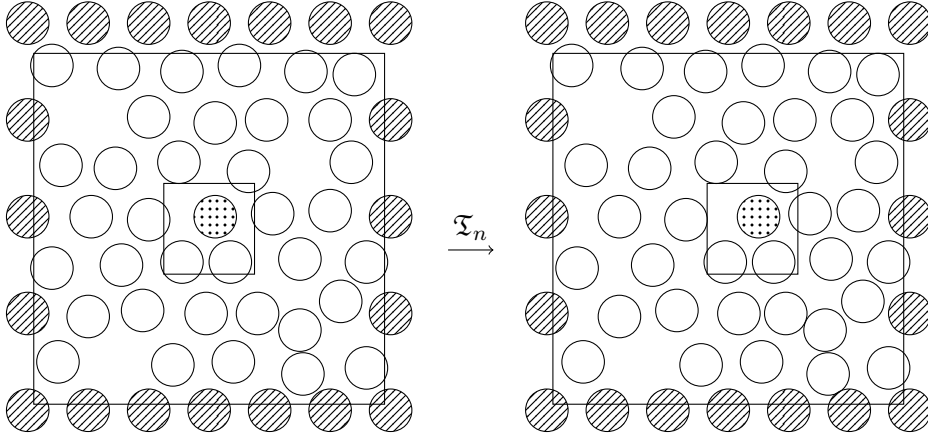


Figure 1: Illustration of the transformation \mathfrak{T}_n . The large square is Λ_n , the small square is of size \sqrt{n} . The disks forming the boundary configuration are shaded. A particle near the center is dotted. The left hand side shows a configuration X , the right hand side shows the corresponding configuration $\mathfrak{T}_n(X)$: Every particle $x \in X$ is moved some distance $t_{n,X}(x)$ to the right. This distance may depend on the initial position x as well as on the part of the configuration X surrounding x .

In the above theorem \mathfrak{T}_n should be thought of a transformation shifting every particle x of a given configuration X by the amount $t_{n,X}(x)$ in direction e_1 , see Figure 1. The middle region $\Lambda_{\sqrt{n}}$ is shifted to the right by $\delta\epsilon\sqrt{\log n}$. The particle density in the left part of Λ_n is decreased and the particle density in the right part of Λ_n is increased. The above properties of the transformation can be interpreted as follows: By (1) the transformation does not affect particles outside of Λ_n , i.e. particles of the boundary configuration. By (2) particles near the origin are shifted by an amount of order $\sqrt{\log n}$ provided the configuration is good. By (3) particles that are close to each other are shifted by almost the same amount. In particular, if particles locally form a lattice, this lattice structure is almost preserved by the transformation. By (4) good configurations are likely. By (5) the transformation is bijective and compatible with the hard core condition. By (6) we have control over probabilities when applying the transformation. By (7) the price to pay when applying the transformation in terms of change of probabilities does not depend on n . The reason why we consider the transformation in some direction along with the corresponding transformation in the opposite direction is that we are not able to obtain the corresponding estimate for $\log \varphi_n$ alone (similar to Mermin-Wagner-type arguments). We note that it is necessary to introduce a set of good configurations, because otherwise the above properties are in conflict: For a configuration with a dense packing of particles to the right of $\Lambda_{\sqrt{n}}$ properties (1) and (6) imply that there is not enough room for particles in $\Lambda_{\sqrt{n}}$ to be moved by the target amount. Configurations like that should be thought of as bad.

Since the hard disk model does not have an a priori lattice structure, it is not clear how to refer to a specific particle and make statements about the variance of its position or the displacement from its ideal lattice position. We would like to argue that in a case like this a theorem as the above is a substitute for these statements. Indeed, Theorem 1 guarantees that it is possible to displace all particles in some region by an amount of order $\sqrt{\log n}$ while preserving local structures, and this can be done without affecting probabilities significantly. The following corollary is meant to motivate this interpretation. It shows that if there is an a posteriori lattice structure by which we may identify some particle $\xi \in X$, whose ideal lattice position is close to the origin, then the mean square deviation of its position from the ideal lattice position is at least of order $\log n$. The proof of this result it will also explain the role of property (7).

Corollary 1 *For $z > 0$, $\epsilon := \min\{\frac{1}{48z}, \frac{1}{4}\}$, $\delta \in (0, \frac{1}{30}]$ and $n \geq 200$ let $\mathfrak{T}_n, \bar{\mathfrak{T}}_n$ be transformations as in Theorem 1. Let $\xi : \mathcal{X}^h \rightarrow \mathbb{R}^2 \cup \{\circ\}$ be a rule for picking a particle from a configuration, i.e. $\xi(X) \in X \cup \{\circ\}$ for all $X \in \mathcal{X}^h$, such that $\mu_n^z(\xi \neq \circ | Y) > \frac{1}{2}$, and such that ξ is compatible with \mathfrak{T}_n in that $\xi(\mathfrak{T}_n(X)) = \xi(X) + t_{n,X}(\xi(X))e_1$ for all $X \in \mathcal{X}^h$ and similarly for $\bar{\mathfrak{T}}_n$. Then for every $\bar{\xi} \in \Lambda_{\sqrt{n}/2}$ and $Y \in \mathcal{X}^h$ we have*

$$\mu_n^z(|\xi - \bar{\xi}| \geq \frac{\delta\epsilon}{2}\sqrt{\log n} | Y) \geq \frac{1}{8} \quad \text{and thus} \quad \mu_n^z(|\xi - \bar{\xi}|^2 1_{\{\xi \neq \circ\}} | Y) \geq \frac{\delta^2 \epsilon^2}{32} \log n.$$

Here $|\cdot|$ denotes the maximum norm. The value \circ of ξ corresponds to a situation where particle ξ is absent in a configuration or can't be chosen unambiguously. While the particle labelling mechanism ξ in the above corollary is quite general, we would like to think of Y forming a perfect triangular lattice with a particle density corresponding to the given value of z , we would like to think of μ_n^z as producing particle configurations that have a lattice structure similar to the one of Y in that most particles can be labelled by corresponding lattice sites in a consistent way. ξ should pick the particle corresponding to the lattice site $\bar{\xi}$. We note that Figure 1 depicts a situation like that. Here $\bar{\xi}$ is the lattice site at the center of the lattice formed by the boundary configuration. The corresponding disk ξ is marked as dotted in both configurations. We stress that it is not clear to which extent the hard core model exhibits a lattice structure. The corollary is merely meant to illustrate that whenever we are able to refer to a particle, the theorem indeed implies a lower bound on the fluctuation of its positions.

We note that we have restricted ourselves to the case of pure hard core repulsion only to simplify the exposition of the proof. In the following we describe various possible generalizations. For details on the corresponding setting and definitions we refer to [R1] and [R2]. Indeed, the proof of the generalizations consists of a combination of the ideas used in this paper and the technical machinery of [R1] and [R2]. Details will be provided in a forthcoming paper. Theorem 1 still holds for the following generalizations:

- U may be nonvanishing outside of the hard core. U still has to be symmetric and translation invariant, and outside of the hard core U has to be smooth with a certain integrability condition on the second derivative (ψ -dominated derivatives for a decay function ψ in the sense of [R1]).
- U may have a hard core of a different shape or no hard core at all. Instead it may have a singularity or be bounded. In addition to the conditions above, here U needs to admit a Ruelle bound (in the sense of [R1]), e.g. U nonnegative or U super-stable and lower regular.
- U may not be smooth outside of the hard core/singularity, but may have discontinuities, and the way U behaves near the hard core/singularity is not relevant. What we need here is a symmetric, translation-invariant potential U that admits a Ruelle bound and is smoothly approximable in the sense of Definition 1 of [R1].
- We may have different types of particles (e.g. hard disks with different radii), or particles with internal degrees of freedoms (e.g. hard squares or hard rods with random orientation). For a formulation of the conditions on U in this case, see Definition 1 of [R2].

3 Fluctuations of particle positions: Corollary 1

Here we deduce Corollary 1 from Theorem 1. Let $z > 0$, $\epsilon := \min\{\frac{1}{48z}, \frac{1}{4}\}$, $\delta \in (0, \frac{1}{30}]$ and $n \geq 200$. Let $\mathfrak{T}_n, \bar{\mathfrak{T}}_n$ transformations as in Theorem 1. Let $\xi : \mathcal{X}^h \rightarrow \mathbb{R}^2 \cup \{\circ\}$ be a rule for picking particles from configurations with the given properties, let $\bar{\xi} \in \Lambda_{\sqrt{n}/2}$ and $Y \in \mathcal{X}^h$. Let

$$E_n := \{|\xi - \bar{\xi}| \geq \frac{\delta\epsilon}{2}\sqrt{\log n}\}, \quad D_n := \{|\xi - \bar{\xi}| < \frac{\delta\epsilon}{2}\sqrt{\log n}\}$$

$$\text{and } D_n^\pm := \{|\xi - \bar{\xi} \pm \delta\epsilon\sqrt{\log ne_1}| < \frac{\delta\epsilon}{2}\sqrt{\log n}\}.$$

All four events tacitly imply $\xi \in \mathbb{R}$, i.e. $\xi \neq \circ$. We note that for all $X \in D_n \cap G_n$ we have $\xi(X) \in \Lambda_{\sqrt{n}}$ since $\bar{\xi} \in \Lambda_{\sqrt{n}/2}$. Thus property (3) of the transformation implies $t_{n,X}(\xi(X)) = \delta\epsilon\sqrt{\log n}$. By the compatibility property of ξ this gives $\xi(\mathfrak{T}_n(X)) = \xi(X) + \delta\epsilon\sqrt{\log ne_1}$, i.e. $\mathfrak{T}_n(X) \in D_n^-$. Thus we have shown that $\mathfrak{T}_n(D_n \cap G_n) \subset D_n^-$. This implies

$$\begin{aligned} \mu_n^z(D_n^-|Y) &\geq \mu_n^z(\mathfrak{T}_n(D_n \cap G_n)|Y) = \int \mu_n^z(dX|Y) 1_{\mathfrak{T}_n(D_n \cap G_n)}(X) \\ &= \int \mu_n^z(dX|Y) 1_{\mathfrak{T}_n(D_n \cap G_n)}(\mathfrak{T}_n(X)) \varphi_n(X) = \int \mu_n^z(dX|Y) 1_{D_n \cap G_n}(X) \varphi_n(X) \end{aligned}$$

using property (4) of the transformation. Combining this with the corresponding estimate for $\bar{\mathfrak{T}}_n$ and the inequality

$$\varphi_n(X) + \bar{\varphi}_n(X) \geq 2\sqrt{\varphi_n(X)\bar{\varphi}_n(X)} \geq 1_{\sqrt{\varphi_n\bar{\varphi}_n} \geq \frac{1}{2}}(X)$$

we obtain

$$\begin{aligned} \mu_n^z(E_n|Y) &\geq \mu_n^z(D_n^- \cup D_n^+|Y) \geq \int \mu_n^z(dX|Y) 1_{D_n \cap G_n}(X) (\varphi_n(X) + \bar{\varphi}_n(X)) \\ &\geq \mu_n^z(D_n \cap G_n \cap \{\sqrt{\varphi_n\bar{\varphi}_n} \geq \frac{1}{2}\}|Y) \\ &\geq \mu_n^z(D_n|Y) - \mu_n^z(G_n^c|Y) - \mu_n^z(\sqrt{\varphi_n\bar{\varphi}_n} \leq \frac{1}{2}|Y). \end{aligned}$$

Property (4) of the transformation implies that $\mu_n^z(G_n^c|Y) \leq \frac{1}{8}$, property (5) implies that

$$\mu_n^z(\sqrt{\varphi_n\bar{\varphi}_n} \leq \frac{1}{2}|Y) \leq \mu_n^z(|\log(\varphi_n\bar{\varphi}_n)| \geq \log 4|Y) \leq \frac{120\delta^2}{\log 4} \leq \frac{1}{8}$$

and by the given properties of ξ we have

$$\mu_n^z(E_n|Y) + \mu_n^z(D_n|Y) = \mu_n^z(\xi \neq \circ|Y) \geq \frac{1}{2},$$

so the above implies

$$\mu_n^z(E_n|Y) \geq \frac{1}{2} - \mu_n^z(E_n|Y) - \frac{1}{8} - \frac{1}{8} \quad \Leftrightarrow \quad \mu_n^z(E_n|Y) \geq \frac{1}{8}.$$

This gives the first estimate and the second estimate is a direct consequence.

4 Transformation: Proof of Theorem 1

Let $z > 0$, $\epsilon := \min\{\frac{1}{48z}, \frac{1}{4}\}$ and $\delta \in (0, \frac{1}{2}]$. These parameters will be fixed throughout this proof and dependencies on these parameters will be suppressed. Let $n \geq 200$. We will construct a corresponding transformation \mathfrak{T}_n and a set of good configurations G_n satisfying properties (1) - (7) from Theorem 1. Let $\tau_n : \mathbb{R} \rightarrow [0, \infty)$ be defined by

$$\tau_n(s) := \begin{cases} \delta\epsilon\sqrt{\log n} & \text{for } s \leq n^{2/3} \\ \frac{3\delta\epsilon}{\sqrt{\log n}}(\log n - \log s) & \text{for } n^{2/3} \leq s \leq n \\ 0 & \text{for } s \geq n. \end{cases} \quad (4.1)$$

We take $t_n^0 := \tau_n(|\cdot|)$ to be a first approximation of the translation distance function $t_{n,X}$. Here $|\cdot|$ denotes the maximum norm on \mathbb{R}^2 . $t_n^0 : \mathbb{R}^2 \rightarrow [0, \infty)$ specifies how far a particle at a given position would like to be shifted in direction e_1 and we call such a function a shift profile, see Figure 2. According to the shift profile t_n^0 every particle is shifted by an amount, which only depends on its position. It can be seen that the corresponding transformation satisfies all properties but (3): Hard cores are not respected since particles that are almost at hard core distance may be shifted by different amounts, which may cause collisions of hard cores. To include property (3), the idea is that every particle should have an effect on all other particles within or close to hard core distance, slowing down the shift of these particles, and thus preventing collisions. This slow down of the shift can be achieved by modifying the shift profile accordingly. The slow down caused by a particle at position $p \in \mathbb{R}^2$ which is

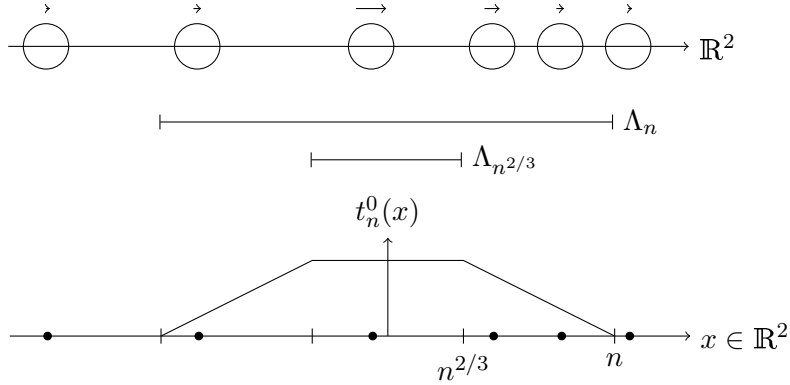


Figure 2: Illustration of the shift profile in a 1D situation. The top part shows a configuration X of disks. The dimensions of the boxes Λ_n and $\Lambda_{n^{2/3}}$ are indicated. For every particle $x \in X$ the translation distance $t_n^0(x)$ is indicated by the arrow above x . The lower part shows t_n^0 in terms of a shift profile. Here the particle positions are indicated by points. The flat part of $t_n^0(x)$ near the origin has height $\delta\epsilon\sqrt{\log n}$. For the sake of the illustration the interpolation (4.1) has been replaced by a linear interpolation.

known to be shifted by the amount $t \geq 0$ can be implemented by a function $m_{p,t} : \mathbb{R}^2 \rightarrow [0, \infty) \cup \{\infty\}$ of the following type:

$$m_{p,t}(x) := \begin{cases} t & \text{for } |x - p|_2 \leq 1 \\ t + \frac{h_{p,t}}{\epsilon}(|x - p|_2 - 1) & \text{for } 1 \leq |x - p|_2 \leq 1 + \epsilon \\ \infty & \text{for } |x - p|_2 > 1 + \epsilon \end{cases}$$

Here

$$h_{p,t} := |\tau_n(|p| - 1 - \epsilon) - t|.$$

is the maximum possible shift amount of a particle at distance $1 + \epsilon$ from p as proposed by τ_n in comparison to the shift amount of the particle at p , see Figure 3. (One should think of the case $\tau_n(|p|) \geq t$.) The parameter $\epsilon = \min\{\frac{1}{48z}, \frac{1}{4}\}$ regulates the range in which the slow down is felt. Functions of the above type will be used to locally modify shift profiles causing slow downs wherever needed. However, these slow downs will prevent collisions only if the slope of these functions can be controlled. So we set

$$m_{p,t}(x) := t \text{ for all } x \in \mathbb{R} \quad \text{if } h_{p,t} > \delta\epsilon. \quad (4.2)$$

For good configurations this proviso will turn out not be necessary. The details of the above construction are chosen to guarantee the following property:

Lemma 1 *Let $p \in \mathbb{R}^2$ and $t \geq 0$. The pointwise minimum of t_n^0 and $m_{p,t}$ is a Lipschitz-continuous function with Lipschitz-constant δ .*

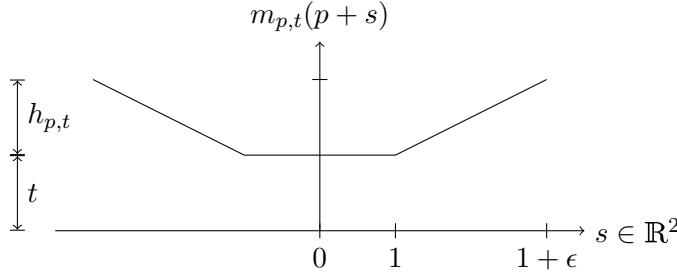


Figure 3: 1D illustration of the function $m_{p,t}$ in case of $h_{p,t} \leq \delta\epsilon$.

Now we are ready to define the transformation $\mathfrak{T}_n : \mathcal{X} \rightarrow \mathcal{X}$. Let $X \in \mathcal{X}$ and $m(X)$ be the number of particles of X_{Λ_n} . We define $\mathfrak{T}_n(X)$ by recursively constructing the following objects for $1 \leq k \leq m(X)$, see Figure 4:

- An enumeration $P_{n,X}^k$ of the particles of X_{Λ_n} .
- Shift amounts $\tau_{n,X}^k \in [0, \infty)$ for the particles $P_{n,X}^k$.
- Shift profiles $t_{n,X}^k : \mathbb{R}^2 \rightarrow [0, \infty)$ that take into account the slow down due to an increasing number of particles.

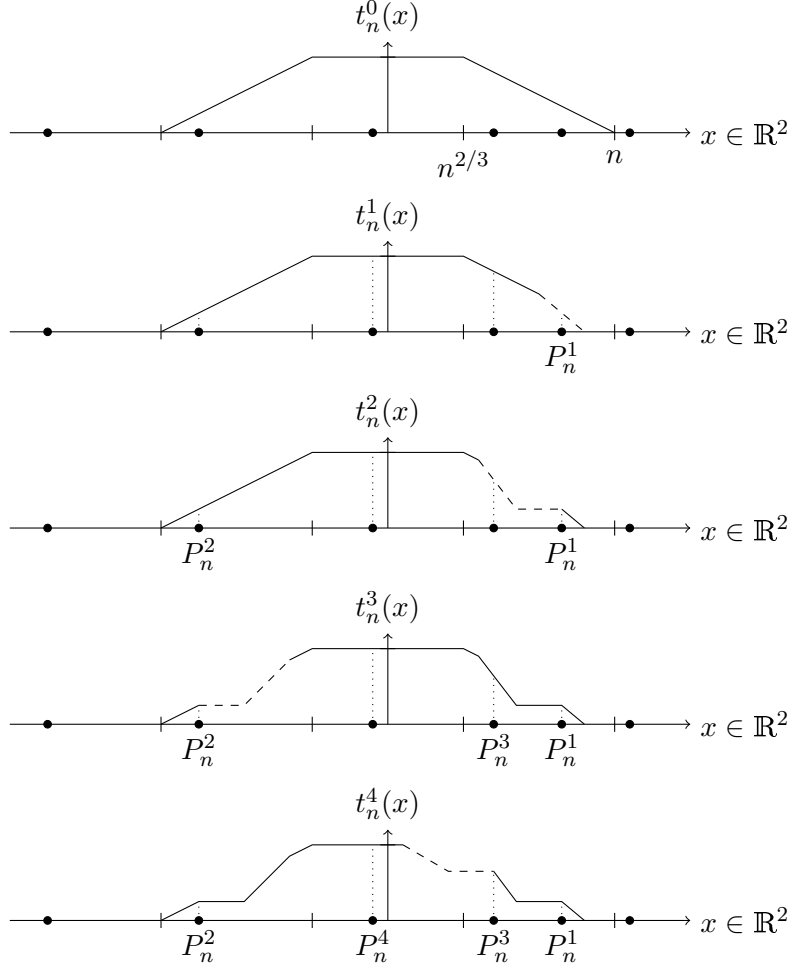


Figure 4: 1D illustration of the recursive construction. The positions of the particles are indicated by dots.

In the above notations we will omit the dependence on X whenever it is clear which configuration X is considered. For fixed $X \in \mathcal{X}$ the k -th step ($1 \leq k \leq m$) of our recursive construction is the following:

- We set the shift profile t_n^k to be the minimum of t_n^{k-1} and the slow down $m_{P_n^{k-1}, \tau_n^{k-1}}$. Here $t_n^0 = \tau_n(|\cdot|)$ as above and $m_{P_n^0, \tau_n^0}$ is the minimum of all slow downs $m_{x,0}$ where $x \in X_{\Lambda_n^c}$.
- Let P_n^k be the point of $X_{\Lambda_n} \setminus \{P_n^1, \dots, P_n^{k-1}\}$ at which the minimum of t_n^k is attained. If there is more than one such point then take the smallest point with respect to the lexicographic order for the sake of definiteness.
- Let $\tau_n^k := t_n^k(P_n^k)$ be the corresponding minimal value of t_n^k .

We also set $\tau_n^0 := 0$ and $|P_n^0| := n$ and think of P_n^0 as an arbitrary point of $X_{\Lambda_n^c}$. The transformation $\mathfrak{T}_n : \mathcal{X} \rightarrow \mathcal{X}$ is now defined by

$$\begin{aligned} \mathfrak{T}_n(X) &:= \{x + t_{n,X}(x)e_1 : x \in X\}, \quad \text{where} \\ t_{n,X}(x) &= 0 \text{ for } x \in X_{\Lambda_n^c} \quad \text{and} \quad t_{n,X}(P_{n,X}^k) = \tau_{n,X}^k \text{ for } 1 \leq k \leq m(X). \end{aligned}$$

For a formulation of suitable properties of the construction we need some more notation. In the above construction $t_n^k(P_n^k)$ is the minimum of $t_n^0(P_n^k)$ and the functions $m_{P_n^l, \tau_n^l}(P_n^k)$ ($0 \leq l < k$). For $0 \leq l < k$ we write

$$P_n^k \rightarrow P_n^l \quad \text{if } t_n^k(P_n^k) = m_{P_n^l, \tau_n^l}(P_n^k),$$

i.e. loosely speaking if the shift of P_n^k is determined by the point P_n^l , and $P_n^k \rightarrow \emptyset$ if $t_n^k(P_n^k) = t_n^0(P_n^k)$, i.e. if the shift of P_n^k does not depend on other points. Let

$$a_{n,X}(P_n^k) := P_n^i, \text{ where } i := \min\{0 \leq l < k : P_n^k \rightarrow \dots \rightarrow P_n^l\},$$

denote the first point of the construction that has an indirect influence on the shift of P_n^k (via a sequence of other points). If the set is empty we set $a_{n,X}(P_n^k) := P_n^k$. We also define

$$m'(X) := \min\{0 \leq l \leq m : h_{P_n^l, \tau_n^l} > \epsilon\delta\}$$

to be the first time in the construction that the proviso (4.2) in the definition of $m_{p,t}$ has to be used. If the set is empty we set $m'(X) := m + 1$.

In the following sequence of lemmas we show that the above construction gives a transformation \mathfrak{T}_n with the desired properties. We start with properties that are almost immediate from the construction.

Lemma 2 *Let $X \in \mathcal{X}$ be a configuration. We have the monotonicity properties:*

$$0 = \tau_n^0 \leq \tau_n^1 \leq \dots \leq \tau_n^m, \quad (4.3)$$

$$t_n^0 \geq t_n^1 \geq \dots \geq t_n^m \geq 0, \quad (4.4)$$

and consequently

$$\forall 0 \leq k \leq l \leq m : t_n(P_n^k) = \tau_n^k = t_n^l(P_n^k). \quad (4.5)$$

Analysing which point has an influence on the shift of which other points we get

$$\forall x \in X : \tau_n(|a_{n,X}(x)|) \leq t_n(x) \leq \tau_n(|x|) \quad (4.6)$$

$$\forall 0 \leq l < k \leq m' : P_n^k \rightarrow P_n^l \Rightarrow |P_n^k - P_n^l|_2 \leq 1 + \epsilon. \quad (4.7)$$

We have the following continuity property:

$$\text{Each } t_n^k \text{ is Lipschitz-continuous with Lipschitz constant } \delta. \quad (4.8)$$

For all particles $x_1, x_2 \in X$ possible hard cores are respected, i.e.

$$|x_1 - x_2|_2 \leq 1 \Rightarrow t_n(x_1) = t_n(x_2), \quad (4.9)$$

$$|x_1 - x_2|_2 > 1 \Rightarrow |(x_1 + t_n(x_1)e_1) - (x_2 + t_n(x_2)e_1)|_2 > 1. \quad (4.10)$$

We note that property (1) of the transformation is satisfied by construction. Property (3) is a consequence of the Lipschitz-continuity (4.8) for t_n^m and $t_n^m = t_n$ from (4.5). The first part of property (5) is shown in the following lemma:

Lemma 3 *The transformation $\mathfrak{T}_n : \mathcal{X} \rightarrow \mathcal{X}$ is bijective.*

Since $\mathfrak{T}_n(\mathcal{X}^h) \subset \mathcal{X}^h$ and $\mathfrak{T}_n((\mathcal{X}^h)^c) \subset (\mathcal{X}^h)^c$ by (4.9) and (4.10), we obtain the second part of property (5). For property (6) we set

$$\varphi_n(X) := \prod_{k=1}^{m(X)} |1 + \partial_{e_1} t_{n,X}^k(P_{n,X}^k)|$$

for $X \in \mathcal{X}$. While proving that this is indeed the density we are looking for, we will also show that this definition makes sense $\mu_n^z(\cdot|Y)$ -a.s., in that the considered partial derivatives exist.

Lemma 4 *For every $Y \in \mathcal{X}^h$ and every measurable function $f \geq 0$*

$$\int d\mu_n^z(dX|Y) f(\mathfrak{T}_n(X)) \varphi_n(X) = \int d\mu_n^z(dX|Y) f(X).$$

This completes the proof of property (6). For property (7) we consider the transformation $\tilde{\mathfrak{T}}_n$, shifting particles by the same amount in the opposite direction. By symmetry it has properties analogous to those of \mathfrak{T}_n . We note that $\tilde{\mathfrak{T}}_n$ is not the inverse of \mathfrak{T}_n . We now show property (7).

Lemma 5 *For every $Y \in \mathcal{X}^h$ we have*

$$\mu_n^z(|\log(\varphi_n \bar{\varphi}_n)||Y) \leq 120\delta^2.$$

The remaining properties (2) and (4) concern good configurations. In light of (4.6), property (2) follows provided we have sufficient control over $a_{n,X}(x)$ for all x . To achieve this control, in light of (4.7) we compare the relation \rightarrow to continuum percolation of disks. For $x, x' \in X_{\Lambda_n}$ we set

$$\begin{aligned} x \sim x' &: \Leftrightarrow |x - x'|_2 \leq 1 + \epsilon \quad \text{and} \\ r_{n,X}(x) &:= \max\{|x'| : x \sim \dots \sim x'\}. \end{aligned}$$

$r_{n,X}(x)$ measures how far the cluster of x reaches. We define

$$G_n := \{X \in \mathcal{X} : \forall x \in X_{\Lambda_n} : r_{n,X}(x) \leq |x| + 3 \log n\}$$

to be the set of all good configurations.

Lemma 6 *For all $X \in G_n$ we have $m'(X) = m(X) + 1$, i.e. the proviso (4.2) in the construction of $\mathfrak{T}_n(X)$ never has to be used, for all $x \in X_{\Lambda_n}$ we have $|a_{n,X}(x)| \leq |x| + 3 \log n + 2$ and in particular*

$$\forall x \in \Lambda_{\sqrt{n}} : t_{n,X}(x) = \delta\epsilon\sqrt{\log n}.$$

This gives property (2). Our choice of ϵ implies that the corresponding continuum percolation is subcritical and thus large clusters have exponentially small probability, so it is reasonable to expect that good configurations in the above sense are not too rare. Indeed we can show property (4):

Lemma 7 *For all $Y \in \mathcal{X}^h$ we have $\mu_n^z(G_n^c|Y) \leq \frac{1}{n}$.*

5 Proof of the lemmas from Subsection 4

5.1 Basic properties: Lemmas 1 and 2

For the proof of Lemma 1 let $p \in \mathbb{R}^2$ and $t \geq 0$. From (4.1) it is easy to see that $t_n^0 = \tau_n(|\cdot|)$ is Lipschitz-continuous with Lipschitz-constant $\frac{3\delta\epsilon}{\sqrt{\log n}} \frac{1}{n^{2/3}} \leq \delta$. By definition every function of the type $m_{p,t}$ is continuous (wherever it is finite) with a slope bounded by δ (thanks to the proviso 4.2). Since the minimum of two Lipschitz-continuous functions is again Lipschitz-continuous with the same Lipschitz-constant, it remains to be shown that

$$\forall x \in \mathbb{R}^2 : |x - p|_2 = 1 + \epsilon \Rightarrow m_{p,t}(x) \geq t_n^0(x)$$

in case of $h_{p,t} \leq \delta\epsilon$. Indeed for $|x - p|_2 = 1 + \epsilon$ by construction

$$m_{p,t}(x) = t + h_{p,t} = t + |\tau_n(|p|) - 1 - \epsilon) - t| \geq \tau_n(|p| - 1 - \epsilon) \geq \tau_n(|x|) = t_n^0(x)$$

since $|p| - 1 - \epsilon \leq |x|$ and τ_n is decreasing. This finishes the proof of Lemma 1.

For the proof of Lemma 2 we use \wedge as notation for the minimum. We consider the construction of $\mathfrak{T}_n(X)$ for a fixed configuration $X \in \mathfrak{X}$. For the first monotonicity property (4.3) we note that for all $1 \leq k \leq m$ we have

$$\tau_n^k = t_n^k(P_n^k) = t_n^{k-1}(P_n^k) \wedge m_{P_n^{k-1}, \tau_n^{k-1}}(P_n^k) \geq \tau_n^{k-1},$$

where the equalities follow from the recursive definition of τ_n^k and t_n^k , and the last step from $t_n^{k-1}(P_n^k) \geq t_n^{k-1}(P_n^{k-1}) = \tau_n^{k-1}$ by definition of P_n^{k-1} and from $m_{p,t} \geq t$. For the second monotonicity property (4.4) it suffices to note that t_n^k is the minimum of t_n^{k-1} and functions of the form $m_{p,t} \geq t$, where $t \geq 0$.

For (4.5) let $0 \leq k \leq l \leq m$. The first equality is by definition, so it suffices to show that $\tau_n^k = t_n^l(P_n^k)$. We note that

$$\forall x \in \mathbb{R}^2 : t_n^l(x) = t_n^k(x) \wedge \bigwedge_{k \leq i < l} m_{P_n^i, \tau_n^i}(x), \quad (5.1)$$

so the above follows from $t_n^k(P_n^k) = \tau_n^k$, which is true by construction and $m_{P_n^i, \tau_n^i} \geq \tau_n^i \geq \tau_n^k$ for all $i \geq k$, which is true by (4.3).

(4.6) is trivial in case of $x \notin \Lambda_n$, since then all three terms equal 0. In case of $x \in \Lambda_n$ we have $x = P_n^k$ for some $k \geq 1$ and $a_{n,X}(P_n^k) = P_n^l$ for some $0 \leq l \leq k$. By definition the latter implies that $P_n^l \rightarrow \emptyset$, i.e. $t_n^0(P_n^l) = t_n^l(P_n^l) = \tau_n^l$. Putting everything together and using (4.3) and (4.4) we see that $t_n(P_n^k) = \tau_n^k$ satisfies

$$\tau_n(|a_{n,X}(P_n^k)|) = t_n^0(P_n^l) = \tau_n^l \leq \tau_n^k = t_n^k(P_n^k) \leq t_n^0(P_n^k),$$

which proves (4.6). (4.7) is immediate from the construction: For $l < k$ such that $P_n^k \rightarrow P_n^l$ we have that $t_n^k(P_n^k) = m_{P_n^l, \tau_n^l}(P_n^k)$ is finite, which implies that $|P_n^k - P_n^l|_2 \leq 1 + \epsilon$ as long as $l < m'$.

The continuity property (4.8) follows from Lemma 1, since t_n^k is the minimum of t_n^0 and functions of the form $m_{p,t}$, and since the minimum of Lipschitz-continuous functions is again Lipschitz continuous.

For the proof of (4.9) and (4.10) let $x_1, x_2 \in X$. Without loss of generality we may suppose that $x_1 = P_n^l$ and $x_2 = P_n^k$, where $0 \leq k < l$. If $|P_n^l - P_n^k|_2 \leq 1$, then using (5.1) for $x = P_n^l$ gives $\tau_n^l = t_n^l(P_n^l) = \tau_n^k$ since $t_n^k(P_n^l) \geq t_n^k(P_n^k) = \tau_n^k$ and $m_{P_n^k, \tau_n^k}(P_n^l) \geq \tau_n^l \geq \tau_n^k$ for all $k \leq i < l$ (and equality in the case $i = k$). Thus we have shown (4.9). Now we consider the case $|P_n^l - P_n^k|_2 > 1$. Let x be the point of the line segment from P_n^l to P_n^k such that $|x - P_n^k|_2 = 1$. From (5.1) we get $t_n^l(x) \leq \tau_n^k$ since $m_{P_n^k, \tau_n^k}(x) = \tau_n^k$. Thus

$$0 \leq \tau_n^l - \tau_n^k \leq t_n^l(P_n^l) - t_n^l(x) \leq \delta |P_n^l - x|_2 = \delta (|P_n^l - P_n^k|_2 - 1)$$

by the Lipschitz-continuity of t_n^l and choice of x . This implies

$$\begin{aligned} |P_n^l + \tau_n^l e_1 - P_n^k - \tau_n^k e_1|_2 &\geq |P_n^l - P_n^k|_2 - |\tau_n^l - \tau_n^k| \\ &\geq |P_n^l - P_n^k|_2 - \delta (|P_n^l - P_n^k|_2 - 1) = (1 - \delta) |P_n^l - P_n^k|_2 + \delta > 1, \end{aligned}$$

which proves (4.10) and thus finishes the proof of Lemma 2.

5.2 Bijectivity of the transformation: Lemma 3

The proof of Lemma 3 can be taken directly from [R1]. For sake of completeness we include a proof here. We have shortened and simplified some of the arguments. We aim to construct $\tilde{\mathfrak{T}}_n$, the inverse transformation with respect to \mathfrak{T}_n . In addition to the objects t_n^k, τ_n^k and P_n^k used in the construction of \mathfrak{T}_n we need the k -step transformation function

$$T_n^k : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T_n^k := id + t_n^k \cdot e_1.$$

By the Lipschitz-continuity of t_n^k , for every $c \in \mathbb{R}$ $T_n^k(., c)$ is continuous and strictly increasing, i.e. for every $0 \leq k \leq m$ we have

$$T_n^k \text{ is bijective and preserves the order of points on horizontal lines.} \quad (5.2)$$

For reference we also note that for all $0 \leq k \leq m$ and $x \in \mathbb{R}^2$ we have

$$(T_n^k)^{-1}(x) + t_n^k((T_n^k)^{-1}(x))e_1 = x. \quad (5.3)$$

For defining an inverse transformation our main task is to reconstruct the enumeration of particles of a configuration when given only the transformed image of the configuration. The following lemma solves this reconstruction problem:

Lemma 8 *Let $X \in \mathcal{X}$, $\tilde{X} := \mathfrak{T}_n(X)$ and $\tilde{P}_n^k := P_n^k + \tau_n^k e_1$ for $1 \leq k \leq m$. For every k \tilde{P}_n^k is the point of $\tilde{X}_{\Lambda_n} \setminus \{\tilde{P}_n^1, \dots, \tilde{P}_n^{k-1}\}$ at which the minimum of $t_n^k \circ (T_n^k)^{-1}$ is attained. If there is more than one such point, then among those \tilde{P}_n^k is the smallest point with respect to lexicographic order.*

Proof: We first show that for all $1 \leq k \leq l$

$$t_n^l \circ (T_n^l)^{-1} \leq t_n^k \circ (T_n^k)^{-1}. \quad (5.4)$$

For a proof let $x \in \mathbb{R}^2$, $x_k := (T_n^k)^{-1}(x)$ and $x_l := (T_n^l)^{-1}(x)$. Both x_k and x_l are to the left of x . Since $t_n^l \leq t_n^k$, $T_n^l(x_k)$ is left of $T_n^k(x_k) = x$. By property

(5.2) for T_n^l this implies that x_k is left of x_l . Using (5.3) this gives $t_n^l(x_l) = |x - x_l| \leq |x - x_k| = t_n^k(x_k)$ and thus proves (5.4). Now let $1 \leq k \leq l \leq m$. By definition we have $t_n^l(P_n^l) = \tau_n^l$, $T_n^l(P_n^l) = \tilde{P}_n^l$, $t_n^k(P_n^k) = \tau_n^k$ and $T_n^k(P_n^k) = \tilde{P}_n^k$. Using (4.4) and (5.4) we deduce

$$t_n^k(T_n^k)^{-1}(\tilde{P}_n^k) = \tau_n^k \leq \tau_n^l = t_n^l(T_n^l)^{-1}(\tilde{P}_n^l) \leq t_n^k(T_n^k)^{-1}(\tilde{P}_n^l).$$

If $t_n^k(T_n^k)^{-1}(\tilde{P}_n^k) = t_n^k(T_n^k)^{-1}(\tilde{P}_n^l)$, then all inequalities in the previous line have to be equalities, i.e. $\tau_n^k = \tau_n^l$ and $\tau_n^l = t_n^l(T_n^l)^{-1}(\tilde{P}_n^l)$. In light of (5.3) this implies that $(T_n^k)^{-1}(\tilde{P}_n^l) = \tilde{P}_n^l - \tau_n^l e_1 = \tilde{P}_n^k$, i.e. $t_n^k(P_n^l) = \tau_n^l = \tau_n^k$. Since lexicographic order is preserved by constant shifts, this concludes the proof of the lemma. \square

The above lemma motivates the following definition of $\tilde{\mathfrak{T}}_n$. Let $\tilde{X} \in \mathcal{X}$ be an arbitrary configuration and $\tilde{m}(\tilde{X})$ be the number of particles of \tilde{X}_{Λ_n} . We recursively define an enumeration $\tilde{P}_{n,\tilde{X}}^k$ of the particles of \tilde{X}_{Λ_n} , shift amounts $\tilde{\tau}_{n,\tilde{X}}^k \in \mathbb{R}$ for the particles $\tilde{P}_{n,\tilde{X}}^k$, shift profiles $\tilde{t}_{n,\tilde{X}}^k : \mathbb{R}^2 \rightarrow [0, \infty)$ and corresponding transformations $\tilde{\mathfrak{T}}_{n,\tilde{X}}^k : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. We then set

$$\begin{aligned} \tilde{\mathfrak{T}}_n(X) &:= \{x - \tilde{t}_{n,\tilde{X}}(x)e_1 : x \in X\}, \quad \text{where} \\ \tilde{t}_{n,\tilde{X}}(x) &= 0 \text{ for } x \in \tilde{X}_{\Lambda_n^c} \quad \text{and} \quad \tilde{t}_{n,\tilde{X}}(\tilde{P}_{n,\tilde{X}}^k) = \tilde{\tau}_{n,\tilde{X}}^k \text{ for } 1 \leq k \leq \tilde{m}(\tilde{X}). \end{aligned}$$

In these notations we again omit the dependence on \tilde{X} whenever it is clear which configuration \tilde{X} is considered. For some fixed configuration $\tilde{X} \in \mathcal{X}$ we now describe the k -th step ($1 \leq k \leq \tilde{m}$) of our recursive construction:

- We set \tilde{t}_n^k to be the minimum of \tilde{t}_n^{k-1} and the slow down $m_{\tilde{P}_n^{k-1} - \tilde{\tau}_n^{k-1}, \tilde{\tau}_n^{k-1}}$. Here $\tilde{t}_n^0 = \tau_n(|\cdot|)$ and $m_{\tilde{P}_n^0 - \tilde{\tau}_n^0, \tilde{\tau}_n^0}$ is the minimum of all $m_{x,0}$ where $x \in \tilde{X}_{\Lambda_n^c}$.
- Let $\tilde{T}_n^k := id + \tilde{t}_n^k \cdot e_1$.
- Let \tilde{P}_n^k be the point of $\tilde{X}_{\Lambda_n} \setminus \{\tilde{P}_n^1, \dots, \tilde{P}_n^{k-1}\}$ at which the minimum of $\tilde{t}_n^k \circ (\tilde{T}_n^k)^{-1}$ is attained. If there is more than one such point then take the smallest point with respect to the lexicographic order.
- Let $\tilde{\tau}_n^k := \tilde{t}_n^k \circ (\tilde{T}_n^k)^{-1}(\tilde{P}_n^k)$ be the corresponding minimal value. Here $\tilde{\tau}_n^0 := 0$.

We need to show that the above construction is well defined, i.e. that \tilde{T}_n^k is invertible in every step, and has suitable monotonicity properties.

Lemma 9 *Let $\tilde{X} \in \mathcal{X}$ and $k \geq 0$. Then*

$$\tilde{T}_n^k \text{ is bijective and preserves the order of points on horizontal lines,} \quad (5.5)$$

$$0 = \tilde{\tau}_n^0 \leq \tilde{\tau}_n^1 \leq \dots \leq \tilde{\tau}_n^m, \quad (5.6)$$

$$\tilde{t}_n^0 \geq \tilde{t}_n^1 \geq \dots \geq \tilde{t}_n^m \geq 0. \quad (5.7)$$

Proof: Whenever \tilde{t}_n^k is well defined, it satisfies the Lipschitz property analogous to (4.8), which gives (5.5). Inductively one can thus show that the above construction is well defined. (5.7) can be show similarly to the corresponding monotonicity property (4.4). For (5.6) we will show $\tilde{\tau}_n^l \geq \tilde{\tau}_n^k$ for every $0 \leq k < l$. Let $x := \tilde{P}_n^l$, $x_k = (\tilde{T}_n^k)^{-1}(x)$, $x_l = (\tilde{T}_n^l)^{-1}(x)$ and $x'_k := x - \tilde{\tau}_n^k e_1$. All x_l, x_k and x'_k are to the left of x . Since $\tilde{t}_n^k(\tilde{T}_n^k)^{-1}(x) \geq \tilde{\tau}_n^k$ by definition of \tilde{P}_n^k , and since we have (5.3) for \tilde{t}_n^k , x_k is to the left of x'_k . By property (5.5) for \tilde{T}_n^k this implies that $\tilde{T}_n^k(x'_k)$ is to the right of $\tilde{T}_n^k(x_k) = x$. Since for \tilde{T}_n^l the shift is slowed down at most to the value $\tilde{\tau}_n^k$ as compared to \tilde{T}_n^k and since $|x - x'_k| = \tilde{\tau}_n^k$, we still have that $\tilde{T}_n^l(x'_k)$ is to the right of x . By (5.5) for \tilde{T}_n^l thus x_l is to the left of x'_k , and thus $\tilde{\tau}_n^l = |x - x_l| \geq |x - x'_k| = \tilde{\tau}_n^k$. \square

In order to show that $\tilde{\mathfrak{T}}_n$ really is the inverse of \mathfrak{T}_n we need an analogue of the reconstruction result from Lemma 8.

Lemma 10 *Let $\tilde{X} \in \mathcal{X}$, \tilde{t}_n^k , \tilde{T}_n^k , \tilde{P}_n^k and $\tilde{\tau}_n^k$ as above, $X := \tilde{\mathfrak{T}}_n(\tilde{X})$ and $P_n^k := \tilde{P}_n^k - \tilde{\tau}_n^k e_1$. For every $1 \leq k \leq \tilde{m}$ P_n^k is the point of $X_{\Lambda_n} \setminus \{P_n^1, \dots, P_n^{k-1}\}$ at which the minimum of \tilde{t}_n^k is attained. If there is more than one such point, then among those P_n^k is the smallest point with respect to lexicographic order.*

Proof: Let $1 \leq k \leq l \leq \tilde{m}$. By (5.3) for \tilde{T}_n^k we have $(\tilde{T}_n^k)^{-1}(\tilde{P}_n^k) = \tilde{P}_n^k - \tilde{\tau}_n^k e_1 = P_n^k$, which implies $\tilde{t}_n^k(P_n^k) = \tilde{\tau}_n^k$. Similarly $\tilde{t}_n^l(P_n^l) = \tilde{\tau}_n^l$. Using (5.6) and (5.7) we obtain

$$\tilde{t}_n^k(P_n^k) = \tilde{\tau}_n^k \leq \tilde{\tau}_n^l = \tilde{t}_n^l(P_n^l) \leq \tilde{t}_n^k(P_n^l).$$

If $\tilde{t}_n^k(P_n^k) = \tilde{t}_n^k(P_n^l)$, then all inequalities in the previous line have to be equalities, so $\tilde{\tau}_n^k = \tilde{\tau}_n^l$ and $\tilde{t}_n^l(P_n^l) = \tilde{t}_n^k(P_n^l)$. This implies $\tilde{P}_n^l = \tilde{T}_n^l(P_n^l) = \tilde{T}_n^k(P_n^l)$, i.e. $(\tilde{T}_n^k)^{-1}(\tilde{P}_n^l) = P_n^l$, which gives $\tilde{t}_n^k((\tilde{T}_n^k)^{-1}(\tilde{P}_n^l)) = \tilde{\tau}_n^l = \tilde{\tau}_n^k$ in light of (5.3). Since lexicographic order is preserved by constant shifts, this concludes the proof of the lemma. \square

Lemma 11 *On \mathcal{X} we have $\tilde{\mathfrak{T}}_n \circ \mathfrak{T}_n = id$ and $\mathfrak{T}_n \circ \tilde{\mathfrak{T}}_n = id$.*

Proof: For the first part let $X \in \mathcal{X}$ and $\tilde{X} := \mathfrak{T}_n(X)$. We have $X_{\Lambda_n^c} = \tilde{X}_{\Lambda_n^c}$ by construction and $\tilde{m}(\tilde{X}) = m(X)$ by (5.2). Now it suffices to prove

$$\tilde{t}_{n,\tilde{X}}^k = t_{n,X}^k, \tilde{T}_{n,\tilde{X}}^k = T_{n,X}^k, \tilde{\tau}_{n,\tilde{X}}^k = \tau_{n,X}^k \text{ and } \tilde{P}_{n,\tilde{X}}^k = P_{n,X}^k + \tau_{n,X}^k \quad (5.8)$$

for every $k \geq 0$ by induction on k . Here $\tilde{P}_{n,\tilde{X}}^0 = P_{n,X}^0 + \tau_{n,X}^0$ is interpreted as $X_{\Lambda_n^c} = \tilde{X}_{\Lambda_n^c}$. The case $k = 0$ is trivial. For the inductive step $k - 1 \rightarrow k$ we observe that $\tilde{t}_n^k = t_n^k$ by induction hypothesis, and $\tilde{T}_n^k = T_n^k$ is an immediate consequence. Combining this with Lemma 8 and the definition of \tilde{P}_n^k we get $\tilde{P}_n^k = P_n^k + \tau_n^k$ and $\tilde{\tau}_n^k = \tau_n^k$.

For the second part let $\tilde{X} \in \mathcal{X}$ and $X := \tilde{\mathfrak{T}}_n(\tilde{X})$. As above it suffices to show (5.8) by induction on k . Here the inductive step follows from Lemma 10. \square

5.3 Density of the transformed process: Lemma 4

Again, the proof of Lemma 4 can be taken directly from [R1]. For sake of completeness we include a proof here. We have shortened and simplified some of the arguments. Let $Y \in \mathcal{X}^h$ and $f \geq 0$ be measurable. By definition of $\mu_n^z(\cdot|Y)$ we have

$$\begin{aligned} & \int d\mu_n^z(dX|Y) f(\mathfrak{T}_n(X)) \varphi_n(X) \\ &= \frac{1}{Z_n^z(Y)} e^{-(2n)^2} \sum_{m \geq 0} z^m \int_{\Lambda_n^m} dx f(\mathfrak{T}_n(Y_x)) \varphi_n(Y_x) 1_{\mathcal{X}^h}(Y_x) \end{aligned}$$

using the shorthand notation $Y_x := \{x_i : i \in J\} \cup Y_{\Lambda_n^c}$ for $x \in \Lambda_n^J$. By (4.9) and (4.10) we have that $Y_x \in \mathcal{X}^h$ iff $\mathfrak{T}_n(Y_x) \in \mathcal{X}^h$ and thus $f(\mathfrak{T}_n(Y_x)) 1_{\mathcal{X}^h}(Y_x) = (f 1_{\mathcal{X}^h})(\mathfrak{T}_n(Y_x))$. Incorporating $1_{\mathcal{X}^h}$ into f it thus suffices to show that

$$\int_{\Lambda_n^m} dx f(\mathfrak{T}_n(Y_x)) \varphi_n(Y_x) = \int_{\Lambda_n^m} dx' f(Y_{x'}) \quad \text{for all } m \geq 0.$$

Since we aim at using the Lebesgue transformation theorem, we would like to enumerate the particles of Y_x , preferably in the same order as they occur in the construction of $\mathfrak{T}_n(Y_x)$. So let Π be the set of all permutations $\eta : \{1, \dots, m\} \rightarrow \{1, \dots, m\}$. For $\eta \in \Pi$ let

$$\begin{aligned} A_\eta &:= \{x \in \Lambda_n^m : \forall 1 \leq k \leq m : x_{\eta(k)} = P_{n,Y_x}^k\} \quad \text{and} \\ \tilde{A}_\eta &:= \{x \in \Lambda_n^m : \forall 1 \leq k \leq m : x_{\eta(k)} = \tilde{P}_{n,Y_x}^k\}, \end{aligned}$$

where \tilde{P}_{n,Y_x}^k are the points from the construction of the inverse transformation in Subsection 5.2. Both A_η and \tilde{A}_η form a disjoint decomposition of Λ_n^m , so it suffices to show that for all $\eta \in \Pi$ we have.

$$\int dx 1_{A_\eta}(x) f(\mathfrak{T}_n(Y_x)) \varphi_n(Y_x) = \int dx' 1_{\tilde{A}_\eta}(x') f(Y_{x'}).$$

By reordering the components of x and x' according to η (and using that a product measure is invariant under permutation of components) it suffices to show the above equality for $\eta = id$. We simplify our notation by setting $A := A_{id}$ and $\tilde{A} := \tilde{A}_{id}$. We now try to express $\mathfrak{T}_n(Y_x)$ as a corresponding transformation $T(x)$. For $x \in \Lambda_n^k$ we define a formal transformation $T(x) := (T_x^k(x_k))_{1 \leq k \leq m}$ recursively by

$$t_x^k := t_x^{k-1} \wedge m_{x_{k-1}, \tau_x^{k-1}}, \quad \tau_x^k := t_x^k(x_k), \quad T_x^k = id + t_x^k \cdot e_1,$$

where $t_x^0 = t_n^0$ and m_{x_0, τ_x^0} is the minimum of all functions $m_{x,0}$ where $x \in Y_{\Lambda_n^c}$. By definition for $x \in A$ we have $x_k = P_{n,Y_x}^k$ for all $1 \leq k \leq m$, which inductively implies that

$$t_{n,Y_x}^k = t_x^k, \quad T_{n,Y_x}^k = T_x^k \quad \text{and} \quad \tau_{n,Y_x}^k = \tau_x^k \quad (5.9)$$

for all $1 \leq k \leq m$ and thus

$$\mathfrak{T}_n(Y_x) = Y_{T(x)}. \quad (5.10)$$

We also note that by construction for every $1 \leq k \leq m$

$$\text{both } t_x^k \text{ and } T_x^k \text{ depend on } x \text{ only through } x_1, \dots, x_{k-1}. \quad (5.11)$$

Furthermore we observe that for all $x \in (\mathbb{R}^2)^k$ we have

$$x \in A \iff T(x) \in \tilde{A}. \quad (5.12)$$

Here “ \Rightarrow ” holds by (5.10) and (5.8) from the proof of Lemma 11. For “ \Leftarrow ” let $x \in (\mathbb{R}^2)^k$ such that $T(x) \in \tilde{A}$ and let $X' := \tilde{\mathfrak{T}}_n(Y_{T(x)})$, where $\tilde{\mathfrak{T}}_n$ is the inverse of \mathfrak{T}_n as defined in the last subsection. By induction

$$\forall 1 \leq k \leq m : \quad T_{n,X'}^k = T_x^k \quad \text{and} \quad x_k = P_{n,X'}^k.$$

In the inductive step $k-1 \rightarrow k$ the first assertion follows from the induction hypothesis and the second follows from the bijectivity of $T_{n,X'}^k$ and

$$T_{n,X'}^k(x_k) = T_x^k(x_k) = \tilde{P}_{n,Y_{T(x)}}^k = P_{n,X'}^k + \tau_{n,X'}^k = T_{n,X'}^k(P_{n,X'}^k),$$

which follows from $T_{n,X'}^k = T_x^k$, the definition of \tilde{A} and (5.8) from the proof of Lemma 11. This completes the proof of the above assertion and we conclude $Y_x = X'$, which implies $x_k = P_{n,X'}^k = P_{n,Y_x}^k$. Thus (5.12) holds. We now get

$$\begin{aligned} \int dx 1_A(x) f(\mathfrak{T}_n(Y_x)) \varphi_n(Y_x) &= \int dx 1_A(x) f(Y_{T(x)}) \prod_{k=1}^m |1 + \partial_{e_1} t_x^k(x_k)| \\ &= \left[\prod_{k=1}^m \int dx_k |1 + \partial_{e_1} t_x^k(x_k)| \right] g(T(x)), \end{aligned}$$

where we have used the definition of φ_n , (5.9) and (5.10) and finally (5.12) using the shorthand notation $g : (\mathbb{R}^2)^k \rightarrow \mathbb{R}$, $g(x) := 1_{\tilde{A}}(x) f(Y_x)$. Now we transform the integrals. For $k = m$ to 1 we substitute $x'_k := T_x^k(x_k)$, making use of (5.11). As before it can be seen that t_x^k is Lipschitz-continuous with Lipschitz-constant $\leq \delta$, so T_x^k is bijective and Lipschitz-continuous and by Rademacher's theorem thus differentiable almost everywhere. Indeed we have

$$\nabla T_x^k = \begin{pmatrix} 1 + \partial_{e_1} t_x^k & \dots \\ 0 & 1 \end{pmatrix} \quad \text{and thus} \quad dx'_k = dx_k |1 + \partial_{e_1} t_x^k(x_k)|$$

for all k by the Lebesgue transformation theorem. So the above integral reduces to

$$\left[\prod_{k=1}^m \int dx'_k \right] g(x') = \int dx' 1_{\tilde{A}}(x') f(Y_{x'}),$$

which finishes the proof of Lemma 4.

5.4 Estimate of the shift amount: Lemma 6

Let $X \in G_n$. We will first show that $h_{P_n^k, \tau_n^k} \leq \delta\epsilon$ for all $0 \leq k \leq m$ by induction on k . For the case $k = 0$ we observe that for every $x \in X_{\Lambda_n^c}$ we have

$$h_{x,0} = \tau_n(|x| - 1 - \epsilon) \leq \tau_n(n - 2) \leq \frac{3\delta\epsilon}{\sqrt{\log n}} \log \frac{n}{n-2} \leq \delta\epsilon.$$

In the inductive step we may assume that $h_{P_n^i, \tau_n^i} \leq \delta\epsilon$ for all $0 \leq i < k$, which implies that $m' \geq k$, and using (4.7) we obtain $|a_{n,X}(P_n^k)| \leq r_{n,X}(P_n^k) + 1 + \epsilon \leq |P_n^k| + 3 \log n + 1 + \epsilon$, where we have also used $X \in G_n$. We thus get

$$\begin{aligned} h_{P_n^k, \tau_n^k} &= \tau_n(|P_n^k| - 1 - \epsilon) - \tau_n^k \leq \tau_n(|P_n^k| - 1 - \epsilon) - \tau_n(|a_{n,X}(P_n^k)|) \\ &\leq \frac{3\delta\epsilon}{\sqrt{\log n} n^{2/3}} (|a_{n,X}(P_n^k)| - |P_n^k| + 1 + \epsilon) \leq \frac{3\delta\epsilon}{\sqrt{\log n} n^{2/3}} (3 \log n + 2.5) \leq \delta\epsilon. \end{aligned}$$

Here we have used the monotonicity of τ_n , the estimate (4.6) for τ_n^k , the upper bound $\frac{3\delta\epsilon}{\sqrt{\log n} n^{2/3}}$ on the derivative of τ_n and $n \geq 200$. This finishes the induction step. In the inductive step we have also shown that $|a_{n,X}(x)| \leq |x| + 3 \log n + 2$. For $x \in X_{\Lambda_{\sqrt{n}}}$ this is bounded by $n^{2/3}$, so (4.6) implies that $t_n(x) = \delta\epsilon\sqrt{\log n}$. This finishes the proof of Lemma 6.

5.5 Strategies for estimating probabilities

In the following sections we need to estimate expectations of sums such as $\sum_{x \in X} f(x)$ with respect to $\mu_n^z(\cdot|Y)$. We present two different strategies for such estimates. The first one relies on the hard core of particles, the second one on properties of the underlying Poisson point process.

Lemma 12 *Let $X \in \mathcal{X}^h$ and $f, g \geq 0$ be measurable functions on \mathbb{R}^2 such that $f(y) \geq g(x)$ for all $y \in B_x := \{x' \in \mathbb{R}^2 : |x' - x|_2 < 1/2\}$. Then*

$$\sum_{x \in X} g(x) \leq \frac{4}{\pi} \int dx f(x). \quad (5.13)$$

Proof: We have $g(x) \leq \frac{4}{\pi} \int_{B_x} dy f(y)$, so

$$\sum_{x \in X} g(x) \leq \sum_{x \in X} \frac{4}{\pi} \int_{B_x} dy f(y) \leq \frac{4}{\pi} \int dy f(y).$$

In the last step we have used that the disks $B_x, x \in X$ are disjoint because of $X \in \mathcal{X}^h$. \square

Lemma 13 *let $Y \in \mathcal{X}^h$, $z > 0$ and $\Lambda \in \mathcal{B}^2$ bounded. Let $g \geq 0$ be measurable on $\Lambda \times \mathcal{X}$. We have*

$$\int \mu_{\Lambda}^z(dX|Y) \sum_{x \in X_{\Lambda}} g(x, X) \leq z \int_{\Lambda} dx \int \mu_{\Lambda}^z(dX|Y) g(x, X \cup \{x\}). \quad (5.14)$$

Proof: (5.14) relies on a corresponding property of the Poisson point process: For Y, z, Λ as above and $f \geq 0$ measurable on $\Lambda \times \mathcal{X}$ we have

$$\int \nu_\Lambda(dX|Y) \sum_{x \in X_\Lambda} f(x, X) = \int \nu_\Lambda(dX|Y) \int_\Lambda dx f(x, X \cup \{x\}). \quad (5.15)$$

To show this, we note that by definition of $\nu_\Lambda(\cdot|Y)$ the left hand side equals

$$e^{-\lambda^2(\Lambda)} \sum_{k \geq 0} \frac{1}{k!} \int_\Lambda dx_1 \dots \int_\Lambda dx_k \sum_{1 \leq l \leq k} f(x_l, \{x_1, \dots, x_k\} \cup Y_{\Lambda^c}).$$

Since the product measure is invariant under permutations, the above equals

$$e^{-\lambda^2(\Lambda)} \sum_{k \geq 1} \frac{k}{k!} \int_\Lambda dx_1 \dots \int_\Lambda dx_k f(x_k, \{x_1, \dots, x_k\} \cup Y_{\Lambda^c}).$$

Substituting $k' := k - 1$, $x := x_k$ and $X = \{x_1, \dots, x_{k-1}\} \cup Y_{\Lambda^c}$ the definition of the Poisson point process implies that the above expression equals the right hand side of (5.15). This finishes the poof of (5.15). Applying (5.15) to the function $f(x, X) := g(x, X)1_{\mathcal{X}^h}(X)z^{\#X_\Lambda}$ and dividing by $Z_\Lambda^z(Y)$ we obtain

$$\int \mu_\Lambda^z(dX|Y) \sum_{x \in X_\Lambda} g(x, X) = \int \mu_\Lambda^z(dX|Y) \int_\Lambda dx g(x, X \cup \{x\})z1_{\{\dots\}},$$

where the indicator enforces that x keeps hard core distance from the particles of X . Estimating this indicator by 1 finishes the proof of (5.14). \square

Multiple sums can be treated by applying the above estimates iteratively. Using Σ^\neq as a shorthand notation for a multiple sum such that the summation indices are assumed to be pairwise distinct we thus get the following:

Lemma 14 *let $Y \in \mathcal{X}^h$, $z > 0$ and $\Lambda \in \mathcal{B}^2$ bounded. Let $f \geq 0$ be measurable on Λ^{m+1} , where $m \geq 1$. We have*

$$\begin{aligned} & \int \mu_\Lambda^z(dX|Y) \sum_{x_1, \dots, x_m \in X_\Lambda, x_0 \in X}^\neq f(x_0, \dots, x_m) \\ & \leq z^m \int \mu_\Lambda^z(dX|Y) \sum_{x_0 \in X} \int_\Lambda dx_1 \dots \int_\Lambda dx_m f(x_0, \dots, x_m). \end{aligned} \quad (5.16)$$

Proof: Applying (5.14) to $g(x_m, X) := \sum^\neq f(x_0, x_1, \dots, x_m)$, where the sum is over all $x_1, \dots, x_{m-1} \in (X \setminus \{x_m\})_\Lambda$ and all $x_0 \in X \setminus \{x_m\}$, we obtain

$$\begin{aligned} & \int \mu_\Lambda^z(dX|Y) \sum_{x_1, \dots, x_m \in X_\Lambda, x_0 \in X}^\neq f(x_0, \dots, x_m) \\ & \leq z \int_\Lambda dx_m \int \mu_\Lambda^z(dX|Y) \sum_{x_1, \dots, x_{m-1} \in X_\Lambda, x_0 \in X}^\neq f(x_0, \dots, x_m). \end{aligned}$$

We note that on the right hand side we have replaced the sum over $x_0 \in (X \cup \{x_m\}) \setminus \{x_m\}$ by a sum over $x_0 \in X$, which doesn't affect the value of the right hand side (and similarly for the sum over x_1, \dots, x_{m-1}). Proceeding inductively we obtain (5.16). \square

5.6 Estimate of the cluster size: Lemma 7

Let $X \in \mathcal{X}^h$. If X is bad, there is a $x \in X_{\Lambda_n}$ such that $r_{n,X}(x) > |x| + 3 \log n$, i.e. there are distinct $x_0, x_1, \dots, x_N \in X_{\Lambda_n}$ such that $x_0 \sim x_1 \sim \dots \sim x_N$ and $N \geq \frac{3}{1+\epsilon} \log n$. Fixing $N := \lceil \frac{3}{1+\epsilon} \log n \rceil$ and introducing the notation

$$A_\epsilon(x) := \{y \in \mathbb{R}^2 : 1 \leq |y - x|_2 \leq 1 + \epsilon\}$$

for an annulus centred at x , the above implies that $x_{i+1} \in A_\epsilon(x_i)$ for all i , and so

$$\begin{aligned} \mu_n^z(G_n^c|Y) &\leq \int \mu_n^z(dX|Y) \sum_{x_0, \dots, x_N \in X_{\Lambda_n}}^{\neq} 1_{\{\forall i: x_{i+1} \in A_\epsilon(x_i)\}} \\ &\leq z^N \int \mu_n^z(dX|Y) \sum_{x_0 \in X_{\Lambda_n}} \int_{\Lambda_n} dx_1 \dots \int_{\Lambda_n} dx_N 1_{\{\forall i: x_{i+1} \in A_\epsilon(x_i)\}} \\ &= z^N \int \mu_n^z(dX|Y) \sum_{x_0 \in X_{\Lambda_n}} \int_{A_\epsilon(x_0)} dx_1 \dots \int_{A_\epsilon(x_{N-1})} dx_N. \end{aligned}$$

Here we have used (5.16). Estimating the integrals using

$$\lambda^2(A_\epsilon(x)) = \pi(2\epsilon + \epsilon^2) \leq \frac{9\pi}{4}\epsilon \leq 8\epsilon,$$

we obtain

$$\begin{aligned} \mu_n^z(G_n^c|Y) &\leq \int \mu_n^z(dX|Y) \sum_{x_0 \in X} 1_{\Lambda_n}(x_0) (8\epsilon z)^N \\ &\leq (8\epsilon z)^N \frac{4}{\pi} \int dx_0 1_{\Lambda_{n+0.5}}(x_0) \leq \frac{1}{6^N} \frac{4}{\pi} (2n+1)^2 \leq \frac{4}{\pi} \frac{(2n+1)^2}{6n^3} \leq \frac{1}{n}. \end{aligned}$$

Here we have used (5.13), the definition of ϵ and the estimate $6^N \geq 6n^3$.

5.7 Estimation of the densities: Lemma 5

Let $X \in \mathcal{X}^h$. We first note that

$$\varphi_n(X) \bar{\varphi}_n(X) = \prod_{k=1}^{m(X)} |1 + \partial_{e_1} t_{n,X}^k(P_{n,X}^k)| \cdot |1 - \partial_{e_1} t_{n,X}^k(P_{n,X}^k)|.$$

By the Lipschitz-continuity from (4.8) we have $|\partial_{e_1} t_{n,X}^k(P_{n,X}^k)| \leq 1/2$. Using $|\log(1-a)| \leq \frac{4}{3}a$ for $0 \leq a \leq 1/4$ we obtain

$$|\log(\varphi_n(X) \bar{\varphi}_n(X))| = \left| \sum_{1 \leq k \leq m} \log(1 - (\partial_{e_1} t_n^k(P_n^k))^2) \right| \leq \sum_{1 \leq k \leq m} \frac{4}{3} (\partial_{e_1} t_n^k(P_n^k))^2.$$

By construction of t_n^k its derivative either equals the derivative of t_n^0 or of one of the functions $m_{P_n^l, \tau_n^l}$ such that $0 \leq l < k$, whenever it exists. Furthermore in case of $k > m'$ its derivative equals 0, since in this case the proviso (4.2) implies

that $t_n^k(x) \leq t_n^k(P_n^k) = \tau_n^{m'}$ for all x . Thus the above can be estimated by the sum of the two following terms:

$$\Sigma_1^n(X) := \sum_{1 \leq k \leq m} \frac{4}{3} (\partial_{e_1} t_n^0(P_n^k))^2, \quad \Sigma_2^n(X) := \sum_{1 \leq k \leq m'} \sum_{0 \leq l < k} \frac{4}{3} (\partial_{e_1} m_{P_n^l, \tau_n^l}(P_n^k))^2$$

For estimating Σ_1^n we use (5.13) to obtain

$$\begin{aligned} \Sigma_1^n(X) &\leq \frac{4}{3} \sum_{x \in X_{\Lambda_n}} (\partial_{e_1} t_n^0(x))^2 \leq \frac{12\delta^2 \epsilon^2}{\log n} \sum_{x \in X} \frac{1_{[n^{2/3}, n]}(|x|)}{|x|^2} \\ &\leq \frac{48\delta^2 \epsilon^2}{\pi \log n} \int dx \frac{1_{[n^{2/3}-0.5, n+0.5]}(|x|)}{|x-0.5|^2} \leq \frac{3\delta^2}{\pi \log n} 3 \log n \leq 3\delta^2. \end{aligned}$$

Here we have used $\epsilon \leq \frac{1}{4}$ and substituted $s := |x|$ for calculating the integral:

$$\begin{aligned} \int dx \frac{1_{[n^{2/3}-0.5, n+0.5]}(|x|)}{(|x|-0.5)^2} &\leq \int_{n^{2/3}-0.5}^{n+0.5} ds \frac{8s}{(s-0.5)^2} \leq \int_{n^{2/3}-1}^n dt \frac{8t+4}{t^2} \\ &\leq 8 \log \frac{n}{n^{2/3}-1} + \frac{4}{n^{2/3}-1} = \frac{8}{3} \log n + 8 \log \frac{n^{2/3}}{n^{2/3}-1} + \frac{4}{n^{2/3}-1} \leq 3 \log n \end{aligned}$$

using $n \geq 200$ in the last step. For Σ_2^n we estimate $|\partial_{e_1} m_{P_n^l, \tau_n^l}(P_n^k)|$ by

$$\begin{aligned} \frac{1}{\epsilon} (\tau_n(|P_n^l| - 1 - \epsilon) - \tau_n^l) &\leq \frac{1}{\epsilon} (\tau_n(|P_n^l| - 1 - \epsilon) - \tau_n(|a_{n,X}(P_n^l)|)) \\ &\leq \frac{3\delta\epsilon c(|P_n^l|)}{\epsilon \sqrt{\log n}} (|a_{n,X}(P_n^l)| - |P_n^l| + 1 + \epsilon), \end{aligned}$$

using (4.6) in the first step and estimating the slope of τ_n in the second step setting

$$c(s) := \frac{1}{\max\{s - 1 - \epsilon, n^{2/3}\}}.$$

We set $k = k_0$, $l = k_1$ and note that we have $m' \geq k_0 > k_1 > \dots > k_N \geq 0$ (with $N \geq 1$) such that for $x_i := P_n^{k_i}$ we have $x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_N = a_{n,X}(x_1) \rightarrow \emptyset$. In particular x_0, \dots, x_N are distinct, $x_0, \dots, x_{N-1} \in \Lambda_n$ and by (4.7) we have $|x_i - x_{i+1}|_2 \leq 1 + \epsilon$ for all i . This gives $|x_N| - |x_1| \leq |x_N - x_1|_2 \leq (1 + \epsilon)(N - 1)$. Treating the cases $N = 1$ and $N \geq 2$ separately, the above implies that $\Sigma_2^n(X)$ can be estimated by the sum of

$$\begin{aligned} \Sigma_{2,1}^n(X) &:= \sum_{x_0 \in X_{\Lambda_n}, x_1 \in X}^{\neq} \frac{12\delta^2(1 + \epsilon)^2}{\log n} c(|x_1|)^2 1_{A_\epsilon(x_1)}(x_0) \quad \text{and} \\ \Sigma_{2,2}^n(X) &:= \sum_{N \geq 2} \sum_{x_0, \dots, x_{N-1} \in X_{\Lambda_n}}^{\neq} \frac{12\delta^2(1 + \epsilon)^2 N^2}{\log n} c(|x_1|)^2 1_{\{\forall i: x_i \in A_\epsilon(x_{i-1})\}}. \end{aligned}$$

Using (5.16) we get

$$\mu_n^z(\Sigma_{2,1}^n|Y) \leq \frac{12(1 + \epsilon)^2 \delta^2 z}{\log n} \int \mu_n^z(dX|Y) \int_{\Lambda_n} dx_0 \sum_{x_1 \in X_{\Lambda_n+1+\epsilon}} c(|x_1|)^2 1_{A_\epsilon(x_1)}(x_0).$$

Now we first estimate $\int_{\Lambda_n} dx_0 1_{A_\epsilon(x_1)}(x_0) \leq \lambda^2(A_\epsilon(x_1)) \leq 8\epsilon$, and then use (5.13) to estimate the sum over x_1 . For this we note that

$$\begin{aligned} \int_{\Lambda_{n+1.5+\epsilon}} dx_1 c(|x_1| - 0.5)^2 &\leq \int_0^{n+1.5+\epsilon} ds \frac{8s}{\max\{(s-1.5-\epsilon)^2, n^{4/3}\}} \\ &= \int_0^{n^{2/3}+1.5+\epsilon} ds \frac{8s}{n^{4/3}} + \int_{n^{2/3}+1.5+\epsilon}^{n+1.5+\epsilon} ds \frac{8s}{(s-1.5-\epsilon)^2} \\ &\leq \frac{4}{n^{4/3}}(n^{2/3}+2)^2 + \int_{n^{2/3}}^n dt \frac{8(t+2)}{t^2} \leq 4(1 + \frac{2}{n^{2/3}})^2 + \int_{n^{2/3}}^n dt \frac{8}{t} + \int_{n^{2/3}}^\infty dt \frac{16}{t^2} \\ &\leq 5 + \frac{8}{3} \log n \leq 4 \log n. \end{aligned}$$

Here we have set $s := |x_1|$ and $t := s - 1.5 - \epsilon$ and used that $n \geq 200$. Thus we finally obtain

$$\mu_n^z(\Sigma_{2,1}^n | Y) \leq \frac{12(1+\epsilon)^2 \delta^2 z}{\log n} \cdot 8\epsilon \cdot \frac{4}{\pi} 4 \log n \leq \frac{192(8\epsilon z)(1+\epsilon)^2 \delta^2}{\pi} \leq 16\delta^2.$$

The expectation $\mu_n^z(\Sigma_{2,2}^n | Y)$ can be estimated similarly. We note that

$$\begin{aligned} \int \mu_n^z(dX | Y) \sum_{x_0, \dots, x_{N-1} \in X_{\Lambda_n}}^{\neq} c(|x_1|)^2 1_{\{\forall i: x_i \in A_\epsilon(x_{i-1})\}} \\ \leq \int \mu_n^z(dX | Y) z^{N-1} \sum_{x_1 \in X_{\Lambda_n}} c(|x_1|)^2 \dots \int_{A_\epsilon(x_{N-2})} dx_{N-1} \int_{A_\epsilon(x_1)} dx_0 \\ \leq \int \mu_n^z(dX | Y) \sum_{x_1 \in X_{\Lambda_n}} c(|x_1|)^2 (8\epsilon z)^{N-1} \leq \frac{4}{\pi} \int_{\Lambda_{n+0.5}} dx_1 c(|x_1| - 0.5)^2 \left(\frac{1}{6}\right)^{N-1} \\ \leq \frac{16 \log n}{\pi} \left(\frac{1}{6}\right)^{N-1}, \end{aligned}$$

first using (5.16) and estimating the arising integrals, then estimating the sum over x_1 using (5.13) and estimating the arising integral as above. Thus

$$\mu_n^z(\Sigma_{2,2}^n | Y) \leq \frac{192\delta^2(1+\epsilon)^2}{\pi} \sum_{N \geq 2} \frac{N^2}{6^{N-1}} \leq 100\delta^2$$

Putting everything together we get

$$\mu_n^z(|\log(\varphi_n \bar{\varphi}_n)| | Y) = (2 + 16 + 100)\delta^2 \leq 120\delta^2.$$

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